**Linear Algebra**

S = Set

x = object in the set

x S = if X is in set of numbers

x s = of x is not in set of numbers

S T = S and T are 2 sets and S is a subset of T

S T = = S and T are 2 sets and S is a subset of T but are not the same

S T = union where x is a member of S or T

S T = intersection where x is a member of S and T

= Set of All real numbers

i, m = rows

j,n= columns

Matrices may be m by n o i by j

r = pivots, non-zero rows

D= pivot columns(dependent columns)

F = non-pivot columns (free variable)

= Complex numbers

a.b = dot product

axb = cross product

**Column multiplication:**

x [1, + y[1,

2, 2,

3] 3]

**Row multiplication:**

x [1,2,3] +

y[1,2,3]

The solution for Ax = B , where matrix \* x = B, gives the point where A meets B.

Length (magnitude)of vector is sqrt (x^2 + y^2)

**Dot product** angle covers length of a going in same direction to length of b = length a \* length b \* cos of theta

**Cross product** covers length of a perpendicular to b. = length a \* length b \* sin of theta

Matrix index, matrix side by side with identity matrix and do reduce row echelon to inverse matrix. A matrix row/col equals identity rows/columns otherwise it is not defined.

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| |  |  |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | --- | --- | | 1 | 2 | | | | 3 | | | | 5 | | 6 | | 7 | | | | 8 | 6 | | 6 | | | | |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | | 1 | | 0 | | 0 | | | 0 | 1 | | 0 | | | 0 | | 0 | | 1 | |   Identity Matrx |

**Upper Triangular determinate** multiplies diagonal digits of a reduced echelon matrix to get determinant

|  |  |  |
| --- | --- | --- |
| 1 | 2 | 3 |
| 0 | 6 | 7 |
| 0 | 0 | 6 |

= 1 x 6 x 6 = 36 (determinant)

**LU Decomposition**

Uses – of multiplier used to get matrix into upper triangular form and places in lower part of Lower triangular form. The lower x upper = original matrix.

**Echelon Form** *(For AX = 0 (null space))****-*** reduces rows to get a linear equation with free variables

**RREF(A)** *(For AX = 0 (null space))* **– Reduced row echelon** form of a matrix has 1 in columns with rest of rows in columns 0. 1 will only 0s to the left. These columns are called pivot columns and are linearly independent. The pivot columns form a bases for column space of a. In below example r1, r2 and r4 form the basis. In this case r1 + r2 + r4 = 0 and is the Null space of the matrix. All other columns are free variables. In this example r3 and r5 are free variables. Any linear combination of the free variables, columns r3 and r5 in this example, can be used to construct vectors of the pivot columns. Note you can swap rows if the a pivot column already has a 0 in it. Pivot columns cannot be 0. Set free variables to 0 to solve for pivot columns.

X particular + Xnull = b

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| 1 | 0 | -1 | 0 | 4 |
| 0 | 1 | 2 | 0 | 1 |
| 0 | 0 | 0 | 1 | -3 |
| 0 | 0 | 0 | 0 | 0 |

The dimension and rank of the above matrix is 3, which are the number of pivot columns The span of the pivot columns should be equal to the span of all vector columns of the matrix.

Scalars – the real number multiplied by the matrix. In example below 3 is the scalar:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| 3 x   |  |  | | --- | --- | | 2 | 1 | | 3 | 7 | |

In Matrix row multiplication you can multiply only if number of columns in matrix A equals the number of rows in the scalar vector B, otherwise it is not defined

Dimension of matrix multiplication will the outer products. EG 3x2 by 2x3 matrix will be a 3x3 using row dimension of the matrix and column dimension of the scalar. # of row of the first matrix by number of columns of the second matrix. Number of rows give number of dimension output landing points , columns gives the number dimension bases starting points.

free variables= null space= f, dimension of column space = Rank = # of pivot columns = n -r

When multiplying Matrix A with Matrix B you multiply Row of A by Columns of B. Multiplication of Matrices are not communitive (order counts A x B is not same as B x A except when multiplying by a 0 matrix) however they are associative (A x (B x C)) = ((A x B) x C). scalar x Transformation x Vector = scalar(T x V)

In Matrix addition the dimensions in the matrix A and the scalar B should be equal otherwise it is not defined.

Matrix \* Vector = 0 is the null space

ij = A ji = A in symmetric matrices

A = identity matrix (Eigen Values/Eigen Values for row space v)

A = Multiplicative inverse (Eigen Vectors/Eigen Values for column space U)

A = is always symmetric

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| |  |  | | --- | --- | | 4 | -3 | | 4 | 3 | | |  |  | | --- | --- | | 4 | 4 | | -3 | 3 | | |  |  | | --- | --- | | 25 | 7 | | 7 | 25 | |

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| |  |  | | --- | --- | | 2 | 3 | | 3 | 4 | | |  |  | | --- | --- | | -4 | 3 | | 3 | -2 | | |  |  | | --- | --- | | 1 | 0 | | 0 | 1 | |

Invert row 1 col 2 with row 2 col 1

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| |  |  | | --- | --- | | 0 | 1 | | 1 | 0 | | |  |  | | --- | --- | | a | b | | c | d | | |  |  | | --- | --- | | a | c | | b | b | |

Row operations have vectors on left, Column operations have vectors on right.

**Determinants:** product of diagonal – product of second diagonal. In below example determinant would be 5x4 – (-1 x4) = 23. A determent of 0 is undefined and means that either the matrix are parallel or they are on the same line.

|  |  |
| --- | --- |
| 5 | 3 |
| -1 | 4 |

**Inverse:** is 1/determinant which in the above case is 1/23 x matrix.

Adjugate: takes the matrix reverse first diagonal and take negative of second diagonal. The adjugate of the above matrix will be:

|  |  |
| --- | --- |
| 4 | -3 |
| 1 | 5 |

EA = I(identity), which means E (elimination matrix) is A inverse, where A A inverse = Identity matrix

EA = U(upper triangle) and A = LU(lower \* upper triangular form)

A is not invertible if A = 0 or if A = B.

The **adjugate** x inverse gives the identity matrix.

LU Decomposition takes a matrix an uses row multiplication to get lower and upper triangular form:

|  |  |  |
| --- | --- | --- |
| Matrix | Lower Triangle | Upper Triag |
| |  |  |  | | --- | --- | --- | | 3 | 6 | 6 | | 3 | 9 | 9 | | 1 | 1 | 3 | | |  |  |  | | --- | --- | --- | | 3 | 7 | 6 | | 8 | 9 | 7 | | 1 | 7 | 4 | | |  |  |  | | --- | --- | --- | | 3 | 7 | 6 | | 8 | 9 | 7 | | 1 | 7 | 4 | |

The **Matrix Rank** is the number of non-zero rows in the reduce row echelon of a Matrix. 3

Full row ranks been r = m and R = I and there is 1 solution for Ax = B

r=n no free variables

n-r with free variables

r = m<n R=[I O] there is 0 or 1 solution for Ax = b

r= n<m R = [I F] infinite solutions for Ax = b

r<m, r<n R=[IF infinite] 0 or infinite solutions for Ax = b

```{r message=FALSE, warning=FALSE

library(matlib)

A<- matrix(c(1,-1,0,5,2,0,1,4,3,1,-2,-2,4,3,1,3),4,4,byrow=FALSE)

A

echelon(A, reduce=FALSE)

A.rank<-R(A)

In this example the rank of matrix A is `r 4

The Maximum rank for an mxn matrix number of linearly independent non-zero m columns in the matrix = n when n<m or = m when n>m. The minimum rank will be the min(m,n). If m<n min rank is m, if n<m min rank is n. Linearly dependent columns are either on same line or go the same direction, never intersecting. Linearly independent columns intersect.

Subspace is the columns with all the linear combinations. Vector space goes through null space or 0,0 coordinates and sub space contains 0 vector.

**Basis for a Space**

Basis for a a space is a sequence of vectors that are independent and span the space. Every basis for a space has the same number of vectors. This is the dimension of the space. Pivot row for vectors written rows give basis of row space in final results. When written in column the pivot column must use original matrix data.

**Eigen vector/ Eigen value - T(v) = λ v**

Eigen value is scalar and eigen vector is a vector

Av = λv for non zero v’s , λ is eigen value of A, if and only if det(λIdentityMatrixV – AV) = 0

Null space of a matrix is the null space of the reduce row echelon form of the matrix where matrix = 0. Means it has Linearly dependent columns, not invertible, non-trivial members in null space, det must be = 0.

Does matrix A x Vector = λ x Vector which equates to eigen value

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| |  |  | | --- | --- | | 3 | 2 | | 3 | -2 | | | |  | | --- | | 2 | | 1 | | = | 4 | |  | | --- | | 2 | | 1 | |

3\*2 + 2\*1 = 8 = 4\*2 = 8

3\*2 + -2 \* 1= 4 4\*1 = 4 iλ of 4 is the eigen value of A any multiple of vector x with be an eigen vector

**SVD – Singular Value Decomposition**

C = U ε

=Vε (Eigen values = of V)

C V = Uε

=

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| |  |  | | --- | --- | | 5 | -1 | | 5 | 7 | | |  |  | | --- | --- | | 5 | 5 | | -1 | 7 | | = | |  |  | | --- | --- | | 26 | 18 | | 18 | 74 | |

Eigen values =det( – λI) = det

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| |  |  | | --- | --- | | 26- λ | 18 | | 18 | 74- λ | | = | -100 λ+1600 | = | (λ-20)( λ-80) |

Eigen vectors = -20, - 80

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| |  |  | | --- | --- | | 6 | 18 | | 18 | 54 | | , V1 =   |  | | --- | | -3/ | | 1/ | |
| |  |  | | --- | --- | | -54 | 18 | | 18 | -6 | | ,V2=   |  | | --- | | 1/ | | 3/ | |
|  |  |
| |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | | V = | |  |  | | --- | --- | | -3/ | 1/ | | 1/ | 3/ | | | | | |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | | ϵ = | |  |  | | --- | --- | |  | 0 | | 0 |  | | |

C V = Uε

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| |  |  | | --- | --- | | 5 | 5 | | -1 | 7 | | |  |  | | --- | --- | | -3/ | 1/ | | 1/ | 3/ | | = | |  |  | | --- | --- | | - |  | |  | 2/ | |

=

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| |  |  | | --- | --- | | -1/ | 1/ | | 1/ | 1/ | | |  |  | | --- | --- | |  | 0 | | 0 |  | |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| U = | |  |  | | --- | --- | | -1/ | 1/ | | 1/ | 1/ | |

**Linear Transformation:**

T(V+W) = T(v) + T(w)

T(cv) = cT(v)

T(cv + dw) = cT(v)+dT(w)

Information need to know T(v) for all inputs for any basis bases v1,v2… vn is T(v1),T(v2)…

Every V = Cv1 + Cv2 +…Cvn

Coordinates come from a basis. **C**oordinates for v = **c**1v1+ …**c**1n1

Construct matrix A that represents linear transformation for T

Choose a basis v1,…vn for inputs of Rn

Choose a basis w1…wn to get output Rm

Projection = P = A/A

Rule to find the Matrix A. Given basis input v1 ---vn and output w1---wn

1st column of A: write T(v1) = a 11w1+a12w2…am1wm

2nd column of A: T(v2)= a 21w1+a22w2…am2wm

Number of permutations in a n x n matrix is n factorial. 4x4 = 4! Or 24 , 3x3= 3! Or 6 permutations

===========================================================

EXERCISES

Exercise C10 Page 310

Find the characteristic polynomial of the matrix A =

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| |  |  |  |  |  |  |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | | det( | |  |  | | --- | --- | | 1 | 2 | | 3 | 4 | | - | |  |  | | --- | --- | | λ | 0 | | 0 | λ | |   ) |

|  |
| --- |
| = (1- λ )\* (4 – λ) – 3 \* 2 = 0 |
| = (1- λ )\* (4 – λ) – 6= 0 |
| = λ^2 +4+4 λ- λ-6 = 0 |
| = (λ^2 -3) (λ-2) = 0 |

Polynomial equation = λ^2 -3 λ-2 = 0

C16 Page 349

Find the matrix representation of:

T ([X Y Z]) =

[3x + 2y + z

x + y + z

x + 3y

2x + 3y + z] =

[3x^2 + 2y^2 + z^2

x^2 + y^2 + z^2

x^2 + 2x^2 + z^2]

**Probability**

Chapter 3 page 88

In a digital computer, a bit is one of the integers {0,1}, and a word is any

string of 32 bits. How many different words are possible?

2 ^ 32 = 4294967296

**Calculus**

F’(x) <-f prim of x-> is the slope of the tangent line to y=f(x0) at the point.

Tangent Line = Limit of Secant Lines PQ as Q --> P. (P is fixed)

Slope of secant is delta f(vertical distance)/delta x (horizontal distance delta = change in)

Slope of tangent is delta line is m = lim as delta x🡪 0

P = f0(fx0)

Q= (x0+delta x, f(x0+delta x)

F’(x0)= m = lim x🡪 0 f(x0+delta x) – f(x0) / delta x {difference quotient)

Example:

F(x) = 1/x = (1/x0 + delta X – 1/x0)/ delta x= -1/xo ^ 2 as delta x 🡪 0

Integration

Integrate by substitution video Lec 30 MIT 1801 Single Variable Calculus

Taylor Series video Lec 38 MIT 1801 Single Variable Calculus